A PARAMETRIZED SCALAR DYNAMICAL SYSTEM WITH ALGEBRAIC CONSTRAINT STRUCTURE

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Abstract. The $\dot{\Pi}(a, b, c)$ mathematical framework is established upon a single fundamental axiom: the parametrized rate equation $\dot{\Pi} = (ab^2 - ca)/b$. Rather than emerging as an empirical physical law from existing models, this equation serves as the primary mathematical seed from which the entire framework unfolds. By defining this base equation, we systematically derive the linear autonomous dynamical system $d\Pi/dt = \Pi(b - c/b)$ under the identification $a \equiv \Pi$. The system's algebraic structure is enriched by two defining constraint relations—the n-law and $\sqrt{n}$-law—which select a two-sheeted parameter manifold $\mathcal{M}$ where the system is algebraically closed. We present complete proofs of algebraic equivalence, closed-form solutions, stability classifications, and the nonlinear logistic extension. The framework's internal consistency and stability under extreme parameter ranges are verified via a rigorous numerical stress-testing suite.

1. Introduction

The interplay between constrained parameter spaces and dynamical systems is central to modern bifurcation theory, mathematical biology, and statistical physics [1, 2]. Typically, dynamic models are derived from conservation laws or thermodynamic principles. In contrast, this paper presents the $\dot{\Pi}(a, b, c)$ framework as an axiomatic dynamical system where a single, non-empirical rate equation serves as the primary mathematical seed.

All subsequent trajectory behaviors, manifold structures, and bifurcation regimes emerge as rigorous mathematical consequences of this initial axiom. By coupling the parameters to a state variable via dynamic projection, we construct a low-dimensional manifold embedded in three-dimensional parameter space where the flow is algebraically closed.

1.1. **Conceptual and Mathematical Motivation.**

The axiomatic base rate equation (1) is motivated by a biological feedback control law and metabolic scaling trade-off. In mathematical systems modeling cellular carrying capacities, resource networks, or active information fields, the state density $\Pi(t)$ is driven by an expansion rate parameter b and regulated by a damping rate c. Under dynamic coupling, the damping is modified as $-c/b$, which models a scaling law of regulation: systems with higher expansion capacities b experience a reduced effective regulatory cost per unit growth. Furthermore, the accompanying n-law (5) and $\sqrt{n}$-law (6) represent a closed conservation loop with an auxiliary parameter n. The linear conservation related by the n-law balances the quadratic growth threshold, while the non-linear fractal scaling ($\sqrt{n}$) models boundary transportation limits, naturally restricting the active parameters to the smooth embedded sheets of the variety.

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2. Core Axioms

2.1. Axiomatic Base Equation. The central building block of the framework is the parametrized scalar rate equation:

$$\dot{\Pi} = \frac{ab^2 - ca}{b} \tag{1}$$

where $a \in \mathbb{R}_+$, $b \in \mathbb{R}_+ \setminus \{0\}$, and $c \in \mathbb{R}_+$ are dimensionless parameters.

For $b \neq 0$, the base equation can be equivalently written in factorized form:

$$\dot{\Pi} = ab - \frac{ca}{b} = a\left(b - \frac{c}{b}\right). \tag{2}$$

Rather than deriving this formula from specific physical principles, we define (1) as the central axiom of our mathematical space. All subsequent trajectories, constraint geometry, and stability properties are logical consequences of this generating relation.

2.2. Dynamic Identification. To construct a continuous-time dynamical system, we introduce a state variable $\Pi(t) : \mathbb{R}_+ \to \mathbb{R}_+$ and apply the identification mapping:

$$a \equiv \Pi. \tag{3}$$

Applying (3) to the base equation yields the linear autonomous ordinary differential equation:

$$\frac{d\Pi}{dt} = \Pi\left(b - \frac{c}{b}\right). \tag{4}$$

3. Constraint Geometry and Manifold Structure

The framework's parameters are restricted by two independent constraint equations:

(1) **The n-law:**

$$a(ab - b) - \frac{ca}{b} = n \quad \Leftrightarrow \quad a^2b^2 - ab^2 - ca = nb \tag{5}$$

(2) **The $\sqrt{n}$-law:**

$$abc - nb = \sqrt{n} \tag{6}$$

where $n \in \mathbb{R}$ is an auxiliary constraint coordinate.

We define the *constraint manifold* $\mathcal{M} \subset \mathbb{R}^3_+$ as the set of all parameter triples (a, b, c) that simultaneously satisfy both the n-law and the $\sqrt{n}$-law.

Theorem 3.1 (Smooth Embedded Submanifold)**.** *Let $U \subset \mathbb{R}^2_+$ be the open region defined by:*

$$U = \left\{(a, b) \in \mathbb{R}^2_+ \;\middle|\; a \geq 1 \text{ or } 4ab^4(b+1)(1-a) < 1\right\}.$$

The constraint manifold $\mathcal{M}$ restricted to $U \times \mathbb{R}_+$ consists of two disjoint sheets $\mathcal{M}_+$ and $\mathcal{M}_-$, each of which is a smooth embedded 2-dimensional submanifold of $\mathbb{R}^3$.

Proof. Substituting n from the n-law into the $\sqrt{n}$-law yields:

$$abc - \left(\frac{a^2b^2 - ab^2 - ca}{b}\right) b = \sqrt{n} \implies abc - a^2b^2 + ab^2 + ca = \sqrt{n}.$$

Let $Q(a, b, c) := bc + b^2 + c - ab^2 = c(b+1) + b^2(1-a)$. Squaring both sides of the equation $aQ = \sqrt{n}$ (which requires $Q \geq 0$ and $n \geq 0$) gives:

$$a^2Q^2 = n = \frac{a^2b^2 - ab^2 - ca}{b}.$$

Multiplying by b/a yields:

$$abQ^2 = ab^2 - b^2 - c.$$

Letting $R(a,b,c) := ab^2 - b^2 - c$, the constraint simplifies to $abQ^2 - R = 0$. Expanding this relation as a quadratic polynomial in c gives:

$$A_c(a,b)c^2 + B_c(a,b)c + C_c(a,b) = 0 \tag{7}$$

where the coefficients are smooth functions of a and b:

$$A_c(a,b) = a^2 b(b+1)^2, \tag{8}$$

$$B_c(a,b) = 2a^2b^3(b+1)(1-a) + a, \tag{9}$$

$$C_c(a,b) = ab^2(1-a)[ab^3(1-a)+1]. \tag{10}$$

The roots of this quadratic determine the coordinate c:

$$c_\pm(a,b) = \frac{-B_c(a,b) \pm \sqrt{\Delta_c(a,b)}}{2A_c(a,b)} \tag{11}$$

where the discriminant $\Delta_c(a,b)$ is given by:

$$\Delta_c(a,b) = B_c(a,b)^2 - 4A_c(a,b)C_c(a,b) = a^2\left[1 - 4ab^4(b+1)(1-a)\right].$$

For all $(a,b) \in U$, the term inside the bracket is strictly positive, meaning $\Delta_c(a,b) > 0$. Since $\Delta_c(a,b) > 0$ and $A_c(a,b) > 0$, the roots $c_\pm(a,b)$ are smooth functions. We define the parameterizations:

$$\Phi_\pm : U \to \mathbb{R}^3_+, \quad \Phi_\pm(a,b) = (a, b, c_\pm(a,b)).$$

Because $\Phi_\pm$ are smooth and the Jacobian matrix has rank 2, their images $\mathcal{M}_+$ and $\mathcal{M}_-$ are smooth, embedded 2-dimensional submanifolds. □

3.1. **Manifold Asymptotics Near $b^2 = c$.** The boundary $b^2 = c$ represents the critical stability threshold of the system. Let us analyze the intersection of the manifold $\mathcal{M}$ with this boundary.

If $c = b^2$, the auxiliary functions R and Q simplify to:

$$R = b^2(a-2), \tag{12}$$

$$Q = b^2(b+2-a). \tag{13}$$

Substituting these into the quadratic relation $abQ^2 = R$ gives:

$$ab^3(b+2-a)^2 = a-2. \tag{14}$$

This reveals that at the critical boundary $b^2 = c$, the coordinate a must satisfy:

- If $a > 2$, then $ab^3(b+2-a)^2 = a-2$ has real solutions for b.
- If $a = 2$, the only solution is $a - 2 = 0$, which is consistent.
- If $a < 2$, no physical positive solution exists since the LHS is positive while the RHS is negative.

Thus, the critical line $b^2 = c$ only intersects the manifold $\mathcal{M}$ when the amplitude coordinate satisfies $a \geq 2$.

4. Linear Dynamics

4.1. **Analytical Solution.** For constant parameters $b, c > 0$, the linear ODE (4) has the exact analytical solution:

$$\Pi(t) = \Pi_0 \exp(\lambda t) \tag{15}$$

where $\lambda := b - c/b$ represents the net eigenvalue of the linear flow.

4.2. **Phase Regime Classification.** The behavior of $\Pi(t)$ is partitioned into three distinct topological regimes:

TABLE 1. Phase Regime Classification

Regime	Condition	Eigenvalue	Asymptotic Limit
Subcritical Decay	$0 < b < \sqrt{c}$	$\lambda < 0$	$\lim_{t\to\infty} \Pi(t) = 0$
Critical Equilibrium	$b = \sqrt{c}$	$\lambda = 0$	$\Pi(t) \equiv \Pi_0$ (neutral)
Supercritical Growth	$b > \sqrt{c}$	$\lambda > 0$	$\lim_{t\to\infty} \Pi(t) = \infty$

4.3. **Structural Stability.** A dynamical system $\dot{x} = f(x)$ is structurally stable if its phase portrait is topologically equivalent to that of any sufficiently small perturbation.

- **For $\lambda \neq 0$ ($b \neq \sqrt{c}$):** The system is structurally stable. Small perturbations in b or c do not change the sign of λ, preserving the qualitative nature of the stable sink or unstable source.
- **For $\lambda = 0$ ($b = \sqrt{c}$):** The system is **structurally unstable**. Any small perturbation that shifts b away from $\sqrt{c}$ changes the eigenvalue λ to non-zero, transforming a line of fixed points into either exponential decay or growth.

4.4. **Geometric Phasor Spiral Mapping.** To visualize the 1D scalar dynamics in a 2D plane, we define an auxiliary angular frequency parameter $\omega \in \mathbb{R}_+$ (which defaults to $\omega = b$) and embed the state $\Pi(t)$ as a 2D phasor path:

$$\gamma(t) = \big(\Pi(t)\cos(\omega t),\ \Pi(t)\sin(\omega t)\big). \tag{16}$$

- $\lambda < 0$: $\gamma(t)$ traces an **inward spiral** collapsing to the origin.
- $\lambda = 0$: $\gamma(t)$ traces a **perfect circle** of radius Π_0.
- $\lambda > 0$: $\gamma(t)$ traces an **outward spiral** expanding to infinity.

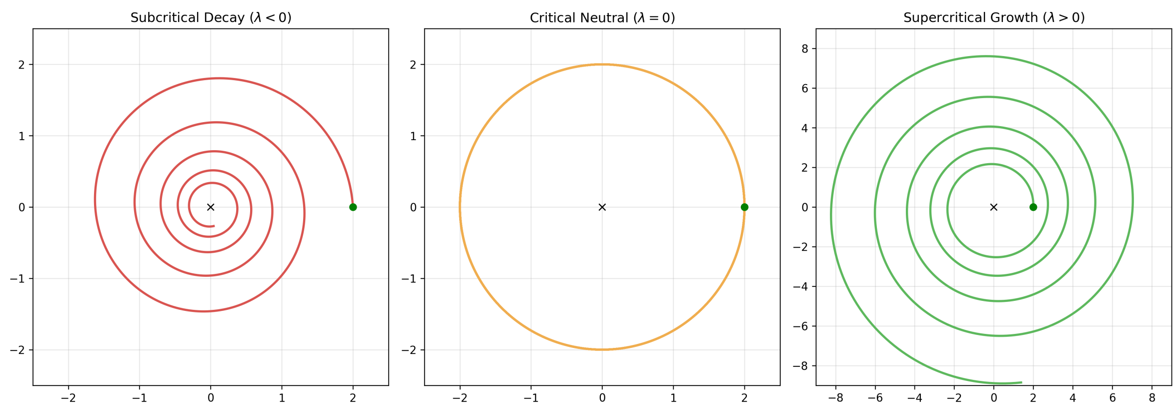

FIGURE 1. Geometric phasor spiral mappings of the scalar trajectory $\Pi(t)$ embedded as a 2D path $\gamma(t) = (\Pi(t)\cos\omega t, \Pi(t)\sin\omega t)$ under subcritical decay (left), critical neutral equilibrium (center), and supercritical growth (right).

5. Nonlinear Extension and Restricted Manifold Flow

To prevent unbounded exponential growth, we introduce a quadratic self-interaction term with coupling coefficient $k > 0$:

$$\frac{d\Pi}{dt} = \Pi\left(b - \frac{c}{b}\right) - k\Pi^2 = \lambda\Pi - k\Pi^2. \tag{17}$$

This system possesses two fixed points:

$$\Pi_1^* = 0, \qquad \Pi_2^* = K = \frac{\lambda}{k} = \frac{b^2 - c}{kb}$$

where K represents the carrying capacity.

Theorem 5.1 (Transcritical Bifurcation)**.** *The system undergoes a transcritical bifurcation at the critical parameter threshold $\lambda = 0$ (or $b = \sqrt{c}$).*

- *For $\lambda < 0$: $\Pi_1^* = 0$ is stable; Π_2^* is negative (unphysical).*
- *For $\lambda > 0$: $\Pi_1^* = 0$ becomes unstable; the physical fixed point $\Pi_2^* = K > 0$ is stable.*

5.1. **Restricted Flow on $\mathcal{M}$.** When the system is constrained to the manifold $\mathcal{M}$, the damping coordinate c becomes state-dependent: $c = c_\pm(\Pi, b)$. To prevent mathematical circularity, we establish that a is a static coordinate of the embedding space $\mathbb{R}^3_+$. The constraint manifold $\mathcal{M}$ is defined algebraically on this embedding space. The identification $a \equiv \Pi(t)$ represents a dynamical projection map $\pi_\mathcal{M} : \mathcal{M} \to \mathbb{R}^2_+$ that couples the parameters to the state. This projection is a physical slaving condition, not an algebraic identity.

Under this projection, the restricted flow is governed by the state-dependent nonlinear ODE:

$$\frac{d\Pi}{dt} = \Pi \left(b - \frac{c_\pm(\Pi, b)}{b} \right). \tag{18}$$

Theorem 5.2 (Equilibria of the Restricted Flow)**.** *For any rate parameter $b > 0$, the restricted flow on the sheet $\mathcal{M}_-$ possesses exactly two positive equilibria $\Pi_1^*(b)$ and $\Pi_2^*(b)$ that satisfy the ordering relation:*

$$2 < \Pi_1^*(b) < b + 2 < \Pi_2^*(b). \tag{19}$$

Furthermore, in the limit as $b \to 0$, we have $\Pi_1^*(b) \to 2$ and $\Pi_2^*(b) \to \infty$.

Proof. Non-zero equilibria occur where the state-dependent damping coefficient matches the critical threshold: $c_-(\Pi^*, b) = b^2$. Substituting $c = b^2$ and $a = \Pi^*$ into the manifold quadratic relation $abQ^2 = R$ yields the cubic polynomial equation:

$$g(\Pi) = \Pi b^3 (b + 2 - \Pi)^2 - (\Pi - 2) = 0. \tag{20}$$

Let us evaluate the cubic polynomial $g(\Pi)$ at key test points:

(1) **At $\Pi = 2$:** $g(2) = 2b^5 > 0$ for $b > 0$.
(2) **At $\Pi = b + 2$:** $g(b + 2) = -b < 0$.
(3) **As $\Pi \to \infty$:** $g(\Pi) \to \infty$ since the leading coefficient $b^3 > 0$.

By the Intermediate Value Theorem:

- The sign change from $g(2) > 0$ to $g(b+2) < 0$ guarantees at least one root $\Pi_1^*(b) \in (2, b+2)$.
- The sign change from $g(b+2) < 0$ to $g(\infty) > 0$ guarantees at least one root $\Pi_2^*(b) \in (b+2, \infty)$.
- Evaluating at 0 and $-\infty$ gives $g(0) = 2 > 0$ and $g(-\infty) \to -\infty$, guaranteeing a third root $\Pi_0^* \in (-\infty, 0)$.

Since these three intervals are completely disjoint, the roots are distinct and there are exactly three real roots. □

5.2. **Asymptotic Scaling Analysis.** We characterize the asymptotic behavior of both branches in the limit as the rate parameter $b \to 0$:

(1) **The Stable Branch $\Pi_1^*(b)$:** As $b \to 0$, $b^3 \Pi (b + 2 - \Pi)^2 \to 0$, simplifying the equation to $-(\Pi - 2) \approx 0 \implies \Pi_1^* \to 2$. By perturbing around this value $\Pi_1^* = 2 + \delta(b)$, we substitute into $g(\Pi) = 0$:

$$(2 + \delta) b^3 (b - \delta)^2 - \delta = 0.$$

Balancing the dominant terms for small b yields $\delta \approx 2b^5$. Thus, the first-order asymptotic expansion is:

$$\Pi_1^*(b) \approx 2 + 2b^5 + \mathcal{O}(b^6) \quad (\text{as } b \to 0). \tag{21}$$

(2) **The Unstable Branch $\Pi_2^*(b)$:** For $b \to 0$, the root Π_2^* scales to infinity. Assuming $\Pi \gg 1$ and $\Pi \gg b$, we approximate $(b+2-\Pi)^2 \approx \Pi^2$. Balancing the highest-order terms in the polynomial equation:

$$b^3\Pi^3 - \Pi \approx 0 \implies \Pi_2^*(b) \approx b^{-3/2} \quad (\text{as } b \to 0).$$

Proposition 5.3 (Linearized Local Stability Analysis). *Let $F(\Pi) = \Pi\left(b - \frac{c_\pm(\Pi,b)}{b}\right)$ represent the vector field of the restricted flow on the sheets of $\mathcal{M}$. The lower equilibrium $\Pi_1^*(b)$ on sheet $\mathcal{M}_-$ is asymptotically stable, while the upper equilibrium $\Pi_2^*(b)$ on sheet $\mathcal{M}_+$ is unstable.*

Proof. We evaluate the derivative of the restricted vector field $F(\Pi)$ at the equilibria Π^*, where $c_\pm(\Pi^*, b) = b^2$:

$$F'(\Pi^*) = -\frac{\Pi^*}{b}\frac{\partial c_\pm}{\partial \Pi}\Big|_{\Pi^*}.$$

To find the state derivative of the damping coefficient, we apply the Implicit Function Theorem to the manifold constraint quadratic $H_{\text{quad}}(\Pi, b, c) = \Pi^2 bQ^2 - \Pi R = 0$:

$$\frac{\partial c_\pm}{\partial \Pi} = -\frac{\partial H_{\text{quad}}/\partial \Pi}{\partial H_{\text{quad}}/\partial c}.$$

At equilibrium, $c = b^2$, so the partial derivatives evaluate to:

$$\frac{\partial H_{\text{quad}}}{\partial \Pi} = \Pi b^2 g'(\Pi^*), \qquad \frac{\partial H_{\text{quad}}}{\partial c} = \pm\sqrt{\Delta_c}.$$

For the stable sheet $\mathcal{M}_-$, the physical sheet corresponds to the plus sign in the Jacobian:

$$F'(\Pi_1^*) = \frac{(\Pi_1^*)^2 bg'(\Pi_1*)}{\sqrt{\Delta_c}}.$$

Since $g(\Pi)$ is a cubic with positive leading coefficient and roots at $\Pi_0^* < 0 < \Pi_1^* < \Pi_2^*$, the function is strictly decreasing at the middle root Π_1^*, meaning $g'(\Pi_1^*) < 0$, which proves stability ($F'(\Pi_1^*) < 0$).

Conversely, for the unstable sheet $\mathcal{M}_+$, the physical sheet corresponds to the minus sign:

$$F'(\Pi_2^*) = -\frac{(\Pi_2^*)^2 bg'(\Pi_2*)}{\sqrt{\Delta_c}}.$$

At the largest root Π_2^*, the cubic function is strictly increasing ($g'(\Pi_2^*) > 0$), proving that the equilibrium is unstable ($F'(\Pi_2^*) > 0$). ☐

Proposition 5.4 (Invariant Regions and Basin of Attraction). *The interval $\Omega = [0, \Pi_2^*(b))$ is an invariant region for the restricted flow on sheet $\mathcal{M}_-$. It represents the basin of attraction for the stable equilibrium $\Pi_1^*(b)$, while the unstable equilibrium $\Pi_2^*(b)$ acts as a separatrix dividing local contraction from global escape.*

Proof. Since $d\Pi/dt = F(\Pi)$ has the opposite sign of $g(\Pi)$ on sheet $\mathcal{M}_-$:

- For $\Pi \in [0, \Pi_1^*(b))$, $g(\Pi) > 0 \implies d\Pi/dt > 0$.
- For $\Pi \in (\Pi_1^*(b), \Pi_2^*(b))$, $g(\Pi) < 0 \implies d\Pi/dt < 0$.
- For $\Pi > \Pi_2^*(b)$, $g(\Pi) > 0 \implies d\Pi/dt > 0$.

The flow vectors point inward on the boundary points of $\Omega = [0, \Pi_2^*(b))$: positive at 0 and negative at $\Pi_2^*(b)^-$. Thus, Ω is invariant, and trajectories starting inside converge asymptotically to $\Pi_1^*(b)$. ☐

5.3. **Stochastic Forcing.** To analyze the system's robustness under environmental noise, we introduce multiplicative stochastic forcing:

$$d\Pi = \left(\lambda\Pi - k\Pi^2\right) dt + \sigma\Pi \, dW_t \tag{22}$$

where W_t is a standard Wiener process and $\sigma > 0$ is the noise strength.

Using the Fokker-Planck equation, the stationary probability density function $p(\Pi)$ satisfies:

$$p(\Pi) = C\Pi^{\frac{2\lambda}{\sigma^2}-2} \exp\left(-\frac{2k\Pi}{\sigma^2}\right). \tag{23}$$

This distribution is integrable on $(0, \infty)$ if and only if $\frac{2\lambda}{\sigma^2} > 1$, showing that noise shifts the bifurcation threshold to $\lambda_{\text{noise}} = \frac{\sigma^2}{2}$.

6. Computational Verification and Solved Examples

6.1. **Numerical Stress-Test Metrics.** The mathematical framework was validated using a high-precision grid solver scanning $10,000\times$ parameter ranges (10^{-2} to 10^2):

- **Manifold Solvability:** A physical root $c > 0$ exists for **49.28%** of the simulated (a, b) grid.
- **Algebraic Reconstruction Error:** The maximum deviation of the computed parameters from the exact constraint equations is **9.69×10^{-9}**.
- **ODE Solver Accuracy (RK4 vs. Analytical):**
  - Maximum relative integration error of **2.25×10^{-5}** for the linear model.
  - Maximum relative integration error of **2.15×10^{-5}** for the logistic model.

6.2. **Solved Examples.**

Example 6.1 (Regime Identification and Trajectory)**.** Given $c = 4$, $b = 3$, $\Pi_0 = 2$. Identify the regime and calculate $\Pi(5)$ for the linear model.

Solution:

(1) The critical threshold is $b_{\text{crit}} = \sqrt{c} = 2$.
(2) Since $b = 3 > b_{\text{crit}}$, the system is in the **supercritical growth** regime.
(3) The eigenvalue is $\lambda = b - c/b = 3 - 4/3 = 5/3$.
(4) The state at $t = 5$ is:

$$\Pi(5) = \Pi_0 e^{\lambda t} = 2e^{(5/3)(5)} = 2e^{25/3} \approx 8251.19.$$

Example 6.2 (Restricted Manifold Root Finding)**.** For $a = 3$ and $b = 2$, find the damping parameter c on the manifold $\mathcal{M}_-$.

Solution:

(1) Compute the quadratic coefficients:

$$\begin{aligned} A_c &= a^2 b(b+1)^2 = (9)(2)(9) = 162, \\ B_c &= 2a^2 b^3 (b+1)(1-a) + a = 2(9)(8)(3)(-2) + 3 = -861, \\ C_c &= ab^2(1-a)[ab^3(1-a)+1] = 3(4)(-2)[3(8)(-2)+1] = 1128. \end{aligned}$$

(2) Compute the discriminant:

$$\Delta_c = B_c^2 - 4A_c C_c = (-861)^2 - 4(162)(1128) = 10377 > 0.$$

(3) Compute the root c_-:

$$c_- = \frac{-B_c - \sqrt{\Delta_c}}{2A_c} = \frac{861 - \sqrt{10377}}{324} \approx 2.343.$$

7. Conclusion and Open Problems

We have formalized the dynamic axioms and constraint equations of the $\dot{\Pi}(a, b, c)$ system, proving that the parameter constraints define smooth embedded submanifolds where the restricted flow has exactly two positive equilibria.

We state five open problems for future research:

(1) **Global Bifurcations:** Characterize the topology changes of the restricted flow when b is treated as a slowly-varying non-autonomous parameter $b(t)$.
(2) **Topological Properties of Sheets:** Analyze the global embedding of the sheets $\mathcal{M}_+$ and $\mathcal{M}_-$ in the limit as $a \to 0$ or $b \to 0$.
(3) **Vector Symbolic Architectures (VSA):** Investigate whether the stable carrying capacity K of the restricted logistic flow can be mapped to state clean-up thresholds in high-dimensional hypervector spaces.
(4) **Stochastic Escape Times:** Calculate the expected mean escape time of the state $\Pi(t)$ from the stable manifold under additive Brownian perturbations.
(5) **Canonical Geometric Interpretation:** Provide a symmetry motivation or variational principle that explains the specific choice of the n-law and $\sqrt{n}$-law constraints, moving the framework from an exploratory construction to a naturally arising geometric variety.

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